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Research Paper

A General type of Liénard Second Order Differential Equation: Classical and Quantum Mechanical Study

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We generate a general model of Liénard type of second order differential equation and study its classical solution. We also study the eigenvalues the Hamiltonian generated from the differential equation.

Keywords: **Liénard Differential Equation; Classical Solution; Phase Portrait; Hamiltonian; Eigenvalues; Matrix Diagonalization Method**

Introduction

Liénard equations are widely used in many branches of science and engineering to model various types of phenomena like oscillations in mechanical and electrical systems. Particularly, for more than fifty years, there has been a continued interest among different authors for paying attention on Liénard type differential equation (Harko *et al.*, 2013; Monsia *et al.*, 2016):

$$\frac{d^2x}{dt^2} + f(x)\left(\frac{dx}{dt}\right)^2 + xg(x) = 0 \tag{1}$$

where $f(x)$ and $g(x)$ are functions of x. Further using suitable choice of $f(x)$ and $g(x)$, one can show that it admits position-dependant mass dynamics and hence will be useful for several applications of quantum physics such as finite gap system (Bravo and Plyushchay, 2016), heterojunctions (Morrow and Brownstein, 1984; Morrow 1987), soliton (Ganguly and Das, 2014), construction of coherent states (Ruby and Senthilvelan, 2010), string physics (Susskind and Uglum, 1996), flux background (Gukov *et al.*, 2004), etc. These types of second order differential equation are interesting for physicists provided one generates suitable Hamiltonian. For all possible values of $f(x)$ and $g(x)$, it may not be possible to generate Hamiltonian having stable eigenvalues. Secondly a classical model solution can also be obtained using He's approximation (He, 2008a; 2008b; Geng and Cai, 2008; Rath, 2011) by using procedure given below

$$\frac{d^2x}{dt^2} + f(x)\left(\frac{dx}{dt}\right)^2 + xg(x) = R(t) \tag{2}$$

Let us consider now two different values of x as

$$x_1 = A\cos\omega_1 t \tag{3}$$

and

$$x_2 = A\cos\omega_2 t \tag{4}$$

then

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$$\omega^2 = \omega_2^2 = \frac{R_2(0)\omega_1^2 - R_1(0)\omega_2^2}{R_2(0) - R_1(0)} \qquad (5)$$

where $\omega_1 = 1$. In this paper, we address the above differential equationby selecting a general type of values on $f(x)$ and $g(x)$, and generate suitable Hamiltonian and study its stable eigenvalues.

General type of Differential Equation and Solution

Here we consider a general type of differential equation as

$$\frac{d^2x}{dt^2} + \frac{N\lambda x^{N-1}}{2\left(1+\lambda x^N\right)}\left(\frac{dx}{dt}\right)^2 + \omega_0^2 \frac{Kx^{K-1}}{2\left(1+\lambda x^N\right)} = 0 \qquad (6)$$

where N, K = 2, 4, 6, ... In this equation one has to fix the value of K and vary N or vice versa. Let us consider the general solution of this differential equation as

$$x = A\cos\omega t \qquad (7)$$

where

$$\omega = \omega_0 \sqrt{\frac{KA^{K-2}}{2(1+\lambda A^N)}} \qquad (8)$$

Hamiltonian Generation

In order to generate Hamiltonians we multiply the differential equation by $\dot{x}$ and rewrite it as

$$\frac{d\left[\frac{\dot{x}^2(1+\lambda x^N)+\omega_0^2 x^K}{2}\right]}{dt} = 0 \qquad (9)$$

Let the bracket term be denoted as H where

$$H = \frac{1}{2}\left[\dot{x}^2(1+\lambda x^N)+\omega_0^2 x^K\right] \qquad (10)$$

Now define momentum, p as

$$p = \frac{\partial H}{\partial \dot{x}} \qquad (11)$$

Hence one can write the Hamiltonian, H as

$$H = \frac{1}{2}\left[\frac{p^2}{(1+\lambda x^N)} + \omega_0^2 x^K\right] \qquad (12)$$

Comparing the above Hamiltonian (Eq. (12)) with $H = \frac{p^2}{2m} + \frac{\omega_0^2 x^K}{2}$, One will find that the mass, m is a function of x i.e., $m(x) = 1+\lambda x^N$. Further, we find an extensive study on the position dependence of mass pertaining to various aspects have been carried out by several authors (von Roos, 1983; Quesne and Tkachuk, 2004; Koç and Tütüncüler, 2003; Dutra *et al.*, 2003; Bagchi *et al.*, 2006; Ganguly *et al.*, 2006; Ganguly and Nieto, 2007; Lévai and Özer, 2010; Killingbeck, 2011; Mazharimousavi, 2012; Yahiaoui and Bentaiba, 2012; Mustafa, 2015; Rajbongshi and Singh, 2015). In the present case, we study the discrete nature of spectra and its stability for the newly generated Hamiltonian (Eqn. (12)) with different values of K and N.

Eigenvalues of Generated Hamiltonian

Here we solve the eigenvalue equation

$$H|\psi\rangle = E|\psi\rangle \qquad (13a)$$

using matrix diagonalization method (Rath *et al.*, 2014; Rath, 2015), in which $|\psi\rangle$ is expressed as

$$|\psi\rangle = \sum_m A_m |m\rangle \qquad (13b)$$

Here $|m\rangle$ satisfy the exact eigenvalue equation

$$H_0|m\rangle = \left(p^2 + x^2\right)|m\rangle = (2m+1)|m\rangle \qquad (14)$$

Now using the above formalism, we get the following recursion relation satisfied by A_m as

$$\sum_{k=2,4,6,....} P_m^k A_{m-k} + S_m A_m + R_m^k A_{m+k} = 0 \qquad (15)$$

where

$$P_m^k = \langle m|H|m-k\rangle \qquad (16a)$$

$$R_m^k = \langle m|H|m+k\rangle \qquad (16b)$$

$$S_m = \langle m | H | m \rangle - E \tag{16c}$$

In fact one will notice that the above Hamiltonian is not invariant under exchange of momentum p and $\frac{1}{(1+\lambda x^N)}$. Hence following the literature (Rath 2008) we write the invariant Hamiltonian as

$$H = \frac{1}{2}\left[p \frac{1}{(1+\lambda x^N)} p + \omega_0^2 x^K \right] \tag{17}$$

and reflect the first four states eigenvalues in Table 1.

Table 1: First four eigenvalues of Hamiltonians with $\omega_0 = 1$, $\lambda = 1$

Hamiltonian	Value of n	Eigenvalue
$H = \frac{1}{2}\left[p \frac{1}{(1+x^2)} p + x^2 \right]$	0	0.355 026 280
	1	1.226 397 537
	2	1.846 999 994
	3	2.445 481 398
$H = \frac{1}{2}\left[p \frac{1}{(1+x^4)} p + x^2 \right]$	0	0.338 179 394
	1	1.199 312 190
	2	1.770 479 342
	3	2.154 962 590
$H = \frac{1}{2}\left[p \frac{1}{(1+x^2)} p + x^4 \right]$	0	0.342 163 615
	1	1.447 762 223
	2	2.733 381 643
	3	3.824 351 590
$H = \frac{1}{2}\left[p \frac{1}{(1+x^4)} p + x^4 \right]$	0	0.326 786 311
	1	1.447 762 223
	2	2.733 381 643
	3	3.824 351 590
$H = \frac{1}{2}\left[p \frac{1}{(1+x^2)} p + x^6 \right]$	0	0.354 476 360
	1	1.652 542 050
	2	3.294 555 429
	3	5.270 061 821
$H = \frac{1}{2}\left[p \frac{1}{(1+x^4)} p + x^6 \right]$	0	0.341 508 635
	1	1.617 435 142
	2	3.393 428 656
	3	5.181 678 146

Phase portrait in the (p, x) plane

Classical phase trajectories of the system (Eqn. (12)) (for $N = K = 2$ and $N = 4$, $K = 2$) are represented in the Figs. 1 and 2 for different parametric choices.

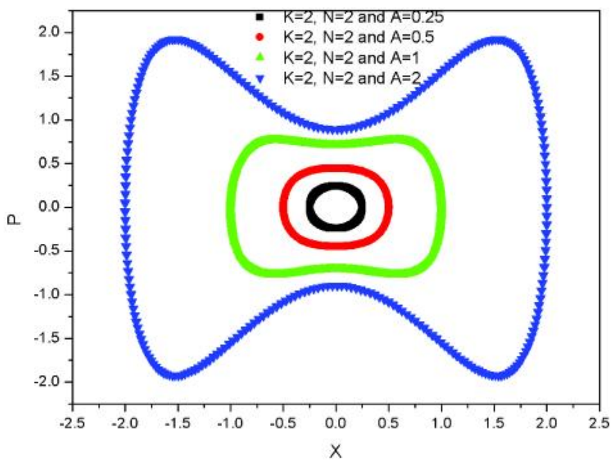

Fig. 1: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, $N = K = 2$, for various values of A

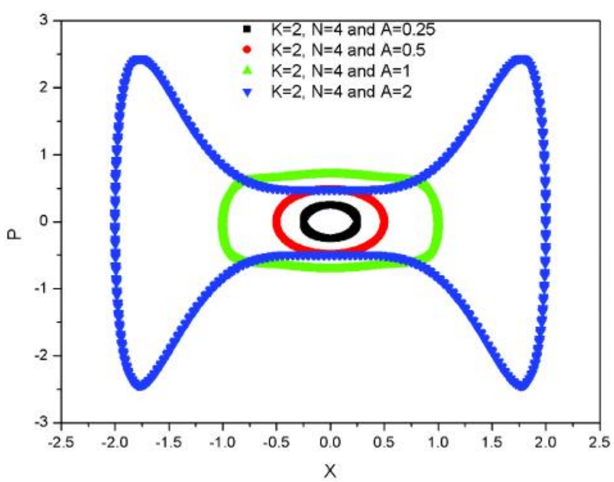

Fig. 2: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, $N = 4$, $K = 2$, for various values of A

Plots reflected in these figures indicate the closed elliptical orbit which clearly justifies the stable behaviour of the system. Similar behaviour has also been reflected for the Hamiltonian with different values of N and K. The quantum mechanical phase trajectories of the system (Eqn. (12)) for $N = K = 2$ and $N = 4$, $K = 2$ are also represented in the Figs. 3 to 4 for different eigenvalues. The representative plot

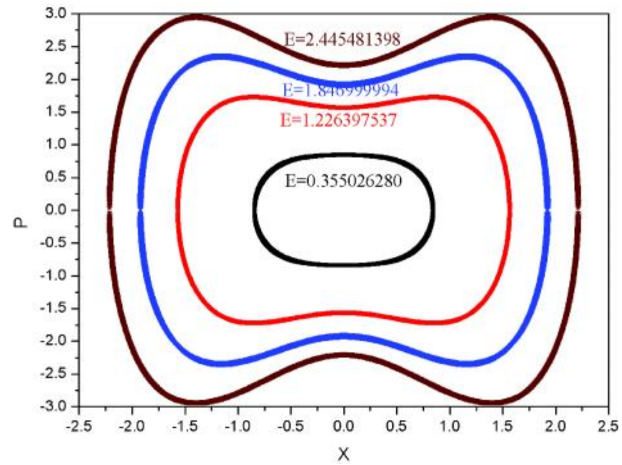

Fig. 3: Phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, $N = K = 2$, for various values of $E = H$

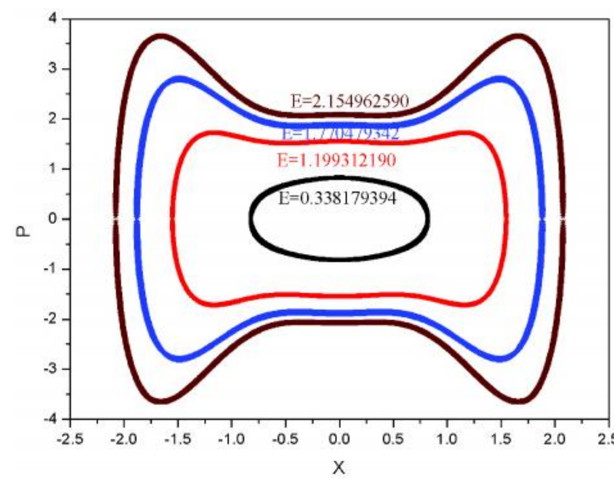

Fig. 4: Phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, $N = 4$, $K = 2$, for various values of $E = H$

$|\psi_n|^2$ of first four eigenstates of Hamiltonian given in eqn. (17) with $N = K = 2$ and $N = 4$, $K = 2$ are shown in Figs. 5 and 6 respectively. We have also seen the similar nature in other Hamiltonians. Here we notice that the nature of phase portraits and corresponding probability are closed orbits thereby justifying the stability of Hamiltonians from both classical and quantum mechanical point of view. At this point, we would like to state that the nature of phase portrait and corresponding probability justify the discrete nature of real spectra and its stability. It is worth mentioning here that the bra state may diverge and its corresponding ket state may converge while the bracket remains the invariant when one study under

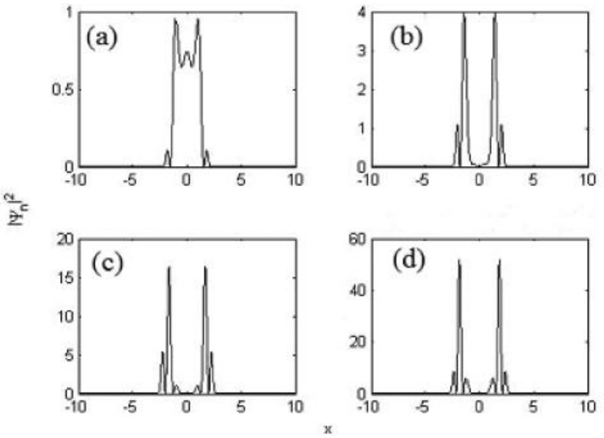

Fig. 5: Wavefunction for the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, $N = K = 2$, for (a) n=0, (b) n=1, (c) n=2 and (d) n=3

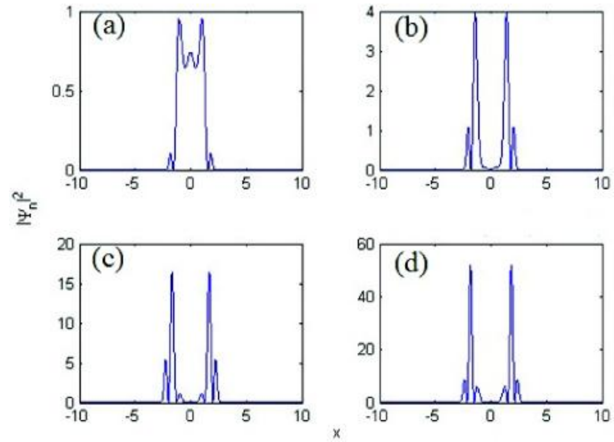

Fig. 6: Wavefunction for the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N =4, K =2, for (a) n=0, (b) n=1, (c) n=2 and (d) n=3

co-ordinate and momentum transformation in complex space (Rath and Mallick, 2016). Classical portrait defined that there exists kinks in the orbit. In order to justify the nature of kinks we plot the probability which is more suitable representation.

Conclusion

In this paper, we formulate a general type of Liénard differential equation which can be regarded as position dependent mass quantum system. We present analytical solution on its classical motion. Classically we also studied the phase portrait for different amplitude of motion and noticed that classically the Hamiltonians are stable. In order to present a complete picture, we calculate eigenstates numerically as the said the equation cannot be solved analytically. The quantum phase portrait has been studied considering E=H. Further, we plot $|\psi_n|^2$ of first four eigenstates and notice that $|\psi_n|^2$ goes to zero as x goes to infinite. This implies that the suggested quantum systems are stable and can yield discrete eigenvalues irrespective of quantum system. We hope to study further on spectral variance under co-ordinate and momentum transformation in complex space (Rath and Mallick, 2016) without changing the nature of probability (Rath, 2017) in a given quantum system.

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